“Bases in spaces of multilinear forms over Banach spaces”

Verónica Dimant e Ignacio Zalduendo

D.T.: N° 2  Septiembre 1994
Introduction

In this note we study the existence of monomial bases in spaces of multilinear forms over Banach spaces. During the past years it has become increasingly apparent that there are close relationships between the existence of monomial bases in spaces of polynomials, the reflexivity of such spaces, and compactness or weak sequential continuity (complete continuity) of polynomials. Thus, for example, for reflexive Banach spaces with bases $E$ and $F$, Holub [17] has proved that reflexivity of $L^2(E \times F)$ is equivalent to compactness of all linear maps from $E$ to $F^*$. Ryan [23] shows that when $E$ and $F$ have shrinking bases, the same condition (compactness of linear operators $E \to F^*$) is equivalent to the existence of a monomial basis in $L^2(E \times F)$. Alencar [2] proved that for reflexive Banach spaces with basis, the existence of monomial basis in the space of homogeneous polynomials is equivalent to reflexivity of this space. Also, the non-existence of monomial basis has been related in some cases ([3], [7], [13]) to the containment of a copy of $\ell^\infty$.

We extend a result in [14] by applying the Gonzalo-Jaramillo indexes and the estimates found in [24] concerning multilinear forms, and use this to give conditions for the existence of monomial basis in $L^m(E_1 \times \cdots \times E_m)$, the space of $m$-linear forms over a product of Banach spaces.

We begin by giving, in §1, a number of conditions including compactness and weak sequential continuity, which are equivalent to the existence of monomial bases. We apply these equivalences and the Gonzalo-Jaramillo indexes to the problem of existence of monomial bases in spaces of multilinear forms over spaces with upper or lower $p$-bounds for sequences in §2, and in spaces of homogeneous polynomials in §3. Some examples, applications and open problems are given in §4.
§1. Multilinear forms and monomial Schauder bases.

Throughout, \( E_1, \ldots, E_m \) will be Banach spaces over the complex field, with Schauder bases \( \{ e_n^1 \}_{n \geq 1}, \ldots, \{ e_n^m \}_{n \geq 1} \) respectively. For each \( i = 1, \ldots, m \), and for each \( n \in \mathbb{N} \), \( e_n^i \) will denote the coordinate functional.

We will denote by \( \mathcal{L}^m(E_1 \times \cdots \times E_m, F) \) the space of all continuous \( m \)-linear mappings from \( E_1 \times \cdots \times E_m \) into \( F \), and by \( \mathcal{L}^m(E_1 \times \cdots \times E_m) \) we mean \( \mathcal{L}^m(E_1 \times \cdots \times E_m, \mathcal{C}) \). We say that \( A \in \mathcal{L}^m(E_1 \times \cdots \times E_m, F) \) is compact if \( A \) maps the product of unit balls \( B_{E_1} \times \cdots \times B_{E_m} \) into a relatively compact subset of \( F \), and that \( A \) is weakly sequentially continuous if \( A \) maps weakly convergent sequences in \( E_1 \times \cdots \times E_m \) into norm convergent sequences in \( F \). The set of all compact \( m \)-linear mappings from \( E_1 \times \cdots \times E_m \) into \( F \) will be denoted by \( \mathcal{K}^m(E_1 \times \cdots \times E_m, F) \) and the set of all weakly sequentially continuous \( m \)-linear mappings will be denoted by \( \mathcal{L}^m_{wsc}(E_1 \times \cdots \times E_m, F) \).

The notation \( E_1 \times \cdots \times E_j \times \cdots \times E_m \) means the \( m-1 \)-product of the spaces \( E_1, \ldots, E_m \), excluding the space \( E_j \). The notation \( A(x_1, \ldots, \hat{x}_j, \ldots, x_m) \) for an \( m-1 \)-linear form \( A \), is interpreted analogously.

Let \( A \in \mathcal{L}^m(E_1 \times \cdots \times E_m) \), \( n \in \mathbb{N} \) and \( 1 \leq k \leq m \), we define
\[
\| A \|^n_k = \sup \{ |A(x_1, \ldots, x_m)| : \| x_i \| = 1 \text{ for all } i \text{ and } x^k \in [e_n^k, e_{n+1}^k, \ldots] \}
\]
and \( \| A \|^n = \sup \{ \| A \|^n_k : k = 1, \ldots, m \} \). Also, if \( T \in \mathcal{L}(E_i, F) \), we write
\[
\| T \|^n = \sup \{ \| T(x) \| : \| x \| = 1 \text{ and } x \in [e_i^1, e_{i+1}^1, \ldots] \}
\]

For each \( (i_1, \ldots, i_m) \in \mathbb{N}^m \) we define \( B_{i_1,\ldots,i_m} \in \mathcal{L}^m(E_1 \times \cdots \times E_m) \) by
\[
B_{i_1,\ldots,i_m} = e_{i_1}^{1*} \cdots e_{i_m}^{m*}
\]

We consider in \( \mathbb{N}^m \) the Square Ordering, described inductively by Ryan in [22] (see also [11]): for \( m = 2 \), \( (1, 1), (1, 2), (2, 2), (2, 1), (1, 3), (2, 3), (3, 3), (3, 2), \ldots \); and given the ordering \( s_1, s_2, \ldots \) of \( \mathbb{N}^{m-1} \), the order in \( \mathbb{N}^m \) is \( (s_1, 1), (s_1, 2), (s_2, 2), (s_2, 1), (s_1, 3), \ldots \). Thus the notation \( \{ B_{i_1,\ldots,i_m} \} \) means the sequence ordered in that way, and \( \sum_{(i_1,\ldots,i_m)\in\mathbb{N}^m} \) means a series with this ordering. On the other hand, \( \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \) refers to an iterated sum.

In [4] \( E \) is said to have the LBP property (for Littlewood, Bogdanowicz, and Pelczynski) if certain analytic functions can be uniformly approximated on bounded sets by polynomials of finite type. Accordingly, we say that \( E_1 \times \cdots \times E_m \) has the \( m \)-LBP property if each \( A \in \mathcal{L}^m(E_1 \times \cdots \times E_m) \) is approximable (uniformly on the product of unit balls) by finite sums of products of linear functionals.

We come now to the main theorem of this section.

**Theorem 1:** Let \( E_1, \ldots, E_m \) be Banach spaces with Schauder bases \( \{ e_n^1 \}, \ldots, \{ e_n^m \} \). Then the following are equivalent:

1. \( E_1, \ldots, E_m \) have the LBP property.
2. \( m \)-linear forms \( \gamma \in \mathcal{L}^m(E_1 \times \cdots \times E_m) \) satisfy the inequality
\[
\sup_{x \in B_{E_1} \times \cdots \times B_{E_m}} \| \gamma(x) \| \leq C \sup_{x \in B_{E_1} \times \cdots \times B_{E_m}} \| x \|^n
\]
for some constant \( C \in \mathbb{R}^+ \).

Conversely, if \( E_1, \ldots, E_m \) have the LBP property and \( \gamma \in \mathcal{L}^m(E_1 \times \cdots \times E_m) \) satisfies the inequality
\[
\sup_{x \in B_{E_1} \times \cdots \times B_{E_m}} \| \gamma(x) \| \leq C \sup_{x \in B_{E_1} \times \cdots \times B_{E_m}} \| x \|^n
\]
for some constant \( C \in \mathbb{R}^+ \), then \( E_1, \ldots, E_m \) have the LBP property.
(i) For all $A \in \mathcal{L}^m(E_1 \times \cdots \times E_m)$, $\|A\|_n \to 0$.

(ii) For all $A \in \mathcal{L}^m(E_1 \times \cdots \times E_m)$, $A = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} A(e_{i_1}^1, \ldots, e_{i_m}^m)B_{i_1, \ldots, i_m}$(uniformly on the product of unit balls).

(iii) For each $i = 1, \ldots, m$, $\{e_n^i\}_{n \geq 1}$ is shrinking and $E_1 \times \cdots \times E_m$ has the $m$-LBP property.

(iv) $\{B_{i_1, \ldots, i_m}\}$ is a Schauder basis of $\mathcal{L}^m(E_1 \times \cdots \times E_m)$.

(v) The basis $\{e_n^i\}_{n \geq 1}$ is shrinking, for all $C \in \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m)$ $\|C\|_n \to 0$ and $\mathcal{L}(E_1, \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m)) = \mathcal{K}(E_1, \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m))$.

(vi) For each $i, j = 1, \ldots, m$, $\{e_n^i\}_{n \geq 1}$ is shrinking and

$$\mathcal{L}^{m-1}(E_1 \times \cdots \times \hat{E}_j \times \cdots \times E_m, E_j^*) = \mathcal{K}^{m-1}(E_1 \times \cdots \times \hat{E}_j \times \cdots \times E_m, E_j^*).$$

(vii) For each $i, j = 1, \ldots, m$, $\{e_n^i\}_{n \geq 1}$ is shrinking and

$$\mathcal{L}^{m-1}(E_1 \times \cdots \times \hat{E}_j \times \cdots \times E_m, E_j^*) = \mathcal{L}^{m-1}_{wsc}(E_1 \times \cdots \times \hat{E}_j \times \cdots \times E_m, E_j^*).$$

(viii) For each $i = 1, \ldots, m$, $\{e_n^i\}_{n \geq 1}$ is shrinking and there is a $j \in \{1, \ldots, m\}$ for which $\mathcal{L}^{m-1}(E_1 \times \cdots \times \hat{E}_j \times \cdots \times E_m, E_j^*) = \mathcal{L}^{m-1}_{wsc}(E_1 \times \cdots \times \hat{E}_j \times \cdots \times E_m, E_j^*)$.

Proof: We first see the equivalence between (i), (ii) and (iii).

(i)⇒(ii) We use induction on $m$. For $m = 1$, the coordinate functionals are a basis of $E^*$ if the basis of $E$ is shrinking, so this is clear. For $m \geq 2$, let $(x^1, \ldots, x^m) \in E_1 \times \cdots \times E_m$, $\|x^i\| = 1$ for all $i$ and let $\varepsilon > 0$. Note that the equality in (ii) is true pointwise:

$$A(x^1, \ldots, x^m) = A(\sum_{i_1=1}^{\infty} e_{i_1}^1(x^1)e_{i_1}^1, \ldots, \sum_{i_m=1}^{\infty} e_{i_m}^m(x^m)e_{i_m}^m)$$

$$= \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} A(e_{i_1}^1, \ldots, e_{i_m}^m)B_{i_1, \ldots, i_m}(x^1, \ldots, x^m)$$
\[ |A(x^1, \ldots, x^m) - \sum_{i_1=1}^{N_1} \cdots \sum_{i_m=1}^{N_m} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ \leq |\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_m=1}^{N_m} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ - \sum_{i_2=1}^{N_2} \sum_{i_m=1}^{N_m} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ + |\sum_{i_1=N_1+1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_m=1}^{N_m} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ = |\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_m=1}^{N_m} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ - \sum_{i_2=1}^{N_2} \sum_{i_m=1}^{N_m} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ + |A(\sum_{i_1=N_1+1}^{\infty} x^1_{i_1} e^1_{i_1}, x^2, \ldots, x^m)| \]

The second term of the last sum is at most \( |A|_{N_1}(1 + K) \) (where \( K \) is the basis constant for \( \{e^1_n\} \)) so we can choose \( N_1 \) so that this term is less than \( \frac{\varepsilon}{2} \). For this \( N_1 \) we can choose \( N_2, \ldots, N_m \) so that the first term of the last sum is less than \( \frac{\varepsilon}{2} \), since if \((i)\) is valid for \( m \)-linear forms then it is valid for \( m - 1 \)-linear forms and we can use the inductive hypothesis.

\((ii)\Rightarrow(iii)\) Clearly \((ii)\) implies that \( E_1 \times \cdots \times E_m \) has the \( m \)-LBP property. In order to see that \( \{e^j_n\}_{n \geq 1} \) is shrinking, let \( \varphi_i \in E^*_i \) and \( A = \varphi_1 \cdots \varphi_m \in L^m(E_1 \times \cdots \times E_m) \). Suppose that for each \( i = 1, \ldots, m, x^i \in E_i, \|x^i\| = 1 \) and \( x^j \in [e^j_n, e^j_{n+1}, \ldots] \) so we have

\[ |A(x^1, \ldots, x^m)| = |\sum_{i_1=1}^{\infty} \sum_{i_m=1}^{\infty} \varphi_1(e^1_{i_1}) \cdots \varphi_m(e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ = |\sum_{i=1}^{\infty} \sum_{i_m=1}^{\infty} \sum_{i_{j=n}}^{\infty} \varphi_1(e^1_{i_1}) \cdots \varphi_m(e^m_{i_m}) B_{i_1,\ldots,i_m}(x^1, \ldots, x^m)| \]

\[ \leq \| \sum_{i=1}^{\infty} \sum_{i_m=1}^{\infty} \sum_{i_{j=n}}^{\infty} \varphi_1(e^1_{i_1}) \cdots \varphi_m(e^m_{i_m}) B_{i_1,\ldots,i_m} \| \]

Since the last of the above terms tends to 0 as \( n \) grows, it follows that \( \|A\|_n^j = \|\varphi_j\|_n \to 0 \). Thus \( \{e^j_n\}_{n \geq 1} \) is shrinking.

\((iii)\Rightarrow(i)\) Let \( A \in L^m(E_1 \times \cdots \times E_m) \) let \( \varepsilon > 0 \). By \((iii)\) there exist \( \varphi^l_1, \ldots, \varphi^l_m \in E^*_i \) such that
In order to prove that $\|A\|^k_n \to 0$, let $M = \max\{\|\varphi^j_i\| : i = 1, \ldots, N; j = 1, \ldots, m\}$ and choose $n_0 \in \mathbb{N}$ such that $\|\varphi^k_i\|_n < \frac{\varepsilon}{2NM^{m-1}}$ for $i = 1, \ldots, N$ and $n \geq n_0$. Suppose that $x^j \in E_j, \|x^j\| = 1$ for $j = 1, \ldots, m$ and $x^k \in [e^k_{n}, e^k_{n+1}, \ldots]$, so we have

$$|A(x^1, \ldots, x^m)| \leq \|(A - \sum_{i=1}^{N} \varphi^1_i \cdot \varphi^m_i)\hat{x}^1(x^1, \ldots, x^m)| + \sum_{i=1}^{N} \varphi^1_i \cdot \varphi^m_i(x^1, \ldots, x^m)|$$

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{N} |\varphi^1_i(x^1)| \cdot |\varphi^m_i(x^m)| < \frac{\varepsilon}{2} + \sum_{i=1}^{N} \frac{\varepsilon}{2NM^{m-1}}M^{m-1} = \varepsilon$$

Thus, $\|A\|^k_n < \varepsilon$ if $n \geq n_0$. Since $k$ is arbitrary, we have shown that $\|A\|_n \to 0$, and this completes the equivalence between the three first items.

Now we prove $(ii) \Rightarrow (iv) \Rightarrow (i)$. Assume that $(ii)$ holds so $\{B_{i_1, \ldots, i_m}\}$ spans a dense subspace in $\mathcal{L}^m(E_1 \times \cdots \times E_m)$. But $B_{i_1, \ldots, i_m}$ are the coordinate functionals corresponding to the canonical basis of the projective tensor product $E_1 \hat{\otimes} \cdots \hat{\otimes} E_m$ (see [22]), so they form a basic sequence. Then $(iv)$ is valid.

Assume, now, that $(iv)$ holds. Let $A \in \mathcal{L}^m(E_1 \times \cdots \times E_m)$ and suppose that $x^j \in E_j, \|x^j\| = 1$ for $j = 1, \ldots, m$ and $x^k \in [e^k_{n}, e^k_{n+1}, \ldots]$. Let $D_k = \{(i_1, \ldots, i_m) \in \mathbb{N}^m : i_k \geq n\}$. By the Square Ordering of $\mathbb{N}^m$, $D_k \subset D = \{(i_1, \ldots, i_m) \in \mathbb{N}^m : (i_1, \ldots, i_m) \geq (1, 1, \ldots, 1, n)\}$. Note also that if $(i_1, \ldots, i_m) \notin D_k$ then $B_{i_1, \ldots, i_m}(x^1, \ldots, x^m) = 0$. Hence,

$$A(x^1, \ldots, x^m) = \sum_{(i_1, \ldots, i_m) \in \mathbb{N}^m} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1, \ldots, i_m}(x^1, \ldots, x^m) = \sum_{(i_1, \ldots, i_m) \in D} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1, \ldots, i_m}(x^1, \ldots, x^m) = \sum_{(i_1, \ldots, i_m) \geq (1, 1, \ldots, 1, n)} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1, \ldots, i_m}(x^1, \ldots, x^m)$$

Therefore,

$$|A(x^1, \ldots, x^m)| \leq \left\| \sum_{(i_1, \ldots, i_m) \geq (1, 1, \ldots, 1, n)} A(e^1_{i_1}, \ldots, e^m_{i_m}) B_{i_1, \ldots, i_m} \right\| \to 0$$
We conclude that $\|A\|_n^k \to 0$ for each $k$ so $\|A\|_n \to 0$ and (i) holds.

We remark that it is clear that (i) implies that all the bases $\{e_n^i\}_{n \geq 1}$ are shrinking and also $\|C\|_n \to 0$ if $C \in \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m)$, so we should prove only the second part of items between (v) and (viii).

Let’s see that (v) holds. Let $T \in \mathcal{L}(E_1, \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m))$ and let $A_T$ be the associated $m$-linear mapping in $\mathcal{L}^m(E_1 \times \cdots \times E_m)$. By (ii),

$$A_T = \sum_{i_1=1}^\infty \cdots \sum_{i_m=1}^\infty A_T(e_{i_1}^1, \ldots, e_{i_m}^m)B_{i_1,\ldots,i_m}$$

so we have

$$T(x^1) = \sum_{i_1=1}^\infty \cdots \sum_{i_m=1}^\infty T(e_{i_1}^1)(e_{i_2}^2, \ldots, e_{i_m}^m)e_{i_1}^1(x^1)e_{i_2}^2 \cdots e_{i_m}^m$$

uniformly for $x^1 \in B_{E_1}$. Thus $T$ is a limit of finite rank operators, and therefore compact.

We now prove that (v)$\Rightarrow$(i). Let $A \in \mathcal{L}^m(E_1 \times \cdots \times E_m)$ and let $T$ be the associated linear mapping in $\mathcal{L}(E_1, \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m))$. Note that for all $n \in \mathbb{N}$,

$$\alpha_n = \sup \{\|A\|_n^k : k = 2, \ldots, m\} = \sup \{\|T(x^1)\|_n : \|x^1\| = 1\}$$

so we will prove that $\alpha_n \to 0$ and that $\|A\|_n^1 \to 0$. Let $\varepsilon > 0$. Since $T$ is compact, there exist $x_1^1, \ldots, x_N^1$ in the unit ball of $E_1$ such that $T(\{x^1 \in E_1 : \|x^1\| = 1\}) \subset \bigcup_{1 \leq j \leq N} B(Tx_j^1, \frac{\varepsilon}{2})$. By (v), for each $x_j^1$ there is an $n_0(x_j^1) \in \mathbb{N}$ such that if $n \geq n_0(x_j^1)$, $\|Tx_j^1\|_n < \frac{\varepsilon}{2}$. Then for $n \geq n_0 = \max \{n_0(x_j^1)\}$ we have that $\|T(x^1)\|_n < \varepsilon$ for all $x^1$ with unit norm. Thus, $\alpha_n \to 0$.

It remains to prove that $\|A\|_n^1 \to 0$. For this, note that $\|A\|_n^1 = \|T\|_n$. Let $\varepsilon > 0$ and take $x_n^1 \in E_1$ such that $\|x_n^1\| = 1$, $x_n^1 \in [e_n^1, e_{n+1}^1, \ldots]$ and $\|T(x_n^1)\| > \|T\|_n - \varepsilon$ for each $n$. Since the basis of $E_1$ is shrinking, we have that $\{x_n^1\}$ converges weakly to 0 so by the compactness of $T$, $T(x_n^1) \to 0$. Therefore, $\|T\|_n \to 0$.

The proof of (ii)$\Rightarrow$(vi)$\Rightarrow$(i) is similar to the proof of (ii)$\Rightarrow$(v)$\Rightarrow$(i).

Finally, we prove (ii)$\Rightarrow$(vii)$\Rightarrow$(viii)$\Rightarrow$(i).

Suppose that (ii) holds. Let $\Phi \in \mathcal{L}^{m-1}(E_1 \times \cdots \times \widehat{E}_j \times \cdots \times E_m, E_j^*)$, for arbitrary $j = 1, \ldots, m$, and take for each $i = 1, \ldots, m, i \neq j$, a sequence $\{x_n^i\}_{n \geq 1}$ converging weakly to $x^i \in E_i$. We may suppose that all these sequences and their limits have norm at most 1. By (ii),

$$\Phi(y^1, \ldots, \widehat{y}^j, \ldots, y^m) = \sum_{i=1}^\infty \cdots \sum_{i_m=1}^\infty \Phi(e_{i_1}^1, \ldots, \widehat{e}_{i_j}^j, \ldots, e_{i_m}^m)(e_{i_j}^j e_{i_1}^1(y_1) \cdots e_{i_m}^m(y^m)e_{i_j}^*)$$

for arbitrary $y^1, \ldots, \widehat{y}^j, \ldots, y^m$.
uniformly in $B_{E_1} \times \cdots \times \hat{B}_{E_j} \times \cdots \times B_{E_m}$. Let $\varepsilon > 0$ and let $N_1, \ldots, N_m$ such that

$$\|\Phi(y^1, \ldots, \hat{y}^j, \ldots, y^m) - \sum_{i_1=1}^{N_1} \cdots \sum_{i_m=1}^{N_m} \Phi(e^1_{i_1}, \ldots, \hat{e}^j_{i_j}, \ldots, e^m_{i_m})(e^1_{i_1}y^1 \cdots e^m_{i_m}y^m)e^j_{i_j}^*| < \frac{\varepsilon}{3}$$

for all $(y^1, \ldots, \hat{y}^j, \ldots, y^m) \in B_{E_1} \times \cdots \times \hat{B}_{E_j} \times \cdots \times B_{E_m}$. Since $x^i_n \to x^i$, we can choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\|\sum_{i_1=1}^{N_1} \cdots \sum_{i_m=1}^{N_m} \Phi(e^1_{i_1}, \ldots, \hat{e}^j_{i_j}, \ldots, e^m_{i_m})(e^1_{i_1}x^1 \cdots e^m_{i_m}x^m)e^j_{i_j}^*| < \varepsilon/3$$

So we have that $\|\Phi(x^1_n, \ldots, x^m_n) - \Phi(x^1, \ldots, x^m)\| < \varepsilon$ for $n \geq n_0$, and $\Phi$ is weakly sequentially continuous.

It is clear that (vii)$\Rightarrow$(viii). We now prove that (viii)$\Rightarrow$(i). Let $A \in \mathcal{L}^m(E_1 \times \cdots \times E_m)$ and let $\Phi$ be the corresponding map in $\mathcal{L}^{m-1}(E_1 \times \cdots \times \hat{E}_j \times \cdots \times E_m, E_j^*)$. Suppose that for some $k$, $\|A\|^k_n \neq 0$, and take $\varepsilon > 0$ and norm-one sequences $\{x^i_n\}_n \in E_i$ for $i = 1, \ldots, m$ such that $x^k_n \to 0$ and $\|A(x^1_n, \ldots, x^m_n)\| > \varepsilon$. We may assume that $\{x^i_n\}_{n \geq 1}$ is weakly Cauchy (for all $i \neq k$), because in a Banach space with a shrinking basis $\{e_n\}$, every bounded sequence $\{x_n\}$ has a weakly Cauchy subsequence. Indeed, $\{e^*_n(x_k)\}_{k \geq 1}$ is Cauchy for all $n$, so the result is a consequence of [6, Lemma 2.7]. We must consider two cases to obtain a contradiction:

a) $k \neq j$: Since $\Phi$ is weakly sequentially continuous we may use [6, Lemma 2.4] (modified for $m$-linear forms from a product of different spaces) to see that

$$\|\Phi(x^1_n, \ldots, \hat{x}^j_n, \ldots, x^k_n, \ldots, x^m_n)\| \to 0$$

Therefore

$$|A(x^1_n, \ldots, x^m_n)| = |\Phi(x^1_n, \ldots, \hat{x}^j_n, \ldots, x^k_n, \ldots, x^m_n)(x^j_n)| \leq \|\Phi(x^1_n, \ldots, \hat{x}^j_n, \ldots, x^k_n, \ldots, x^m_n)\| \to 0$$

and this is a contradiction.

b) $k = j$: By [6, Corollary 2.5], $\Phi$ maps weakly Cauchy sequences into norm Cauchy sequences. Thus $\Phi(x^1_n, \ldots, \hat{x}^k_n, \ldots, x^m_n)$ is a norm Cauchy sequence in $E^*_k$, so it converges to $z \in E^*_k$, and $\|z\|_n \to 0$ because the basis of $E_k$ is shrinking. Hence,

$$|A(x^1_n, \ldots, x^m_n)| = |\Phi(x^1_n, \ldots, \hat{x}^k_n, \ldots, x^m_n)(x^k_n)| \leq \|\Phi(x^1_n, \ldots, \hat{x}^k_n, \ldots, x^m_n)\| \to 0$$

and this is also a contradiction.
Again, we have a contradiction, and this completes the proof.

We end this section with two corollaries and a few remarks.

**Corollary 1:** If the Banach spaces $E_1, \ldots, E_m$ all have unconditional shrinking bases, then either

a) monomials form a basis of $\mathcal{L}^m(E_1 \times \cdots \times E_m)$.

or b) $\mathcal{L}^m(E_1 \times \cdots \times E_m)$ contains a copy of $\ell^\infty$.

Proof: If a) does not hold, by v) of Theorem 1 either there is a non-compact $T \in \mathcal{L}(E_1, \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m))$, or a $C \in \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m)$ with $\|C\|_n$ not tending to zero. In the first case, $\mathcal{L}(E_1, \mathcal{L}^{m-1}(E_2 \times \cdots \times E_m))$ contains $\ell^\infty$ by a result of Diestel and Morrison ([8], Theorem 2). In the second, by i) of Theorem 1 monomials do not form a Schauder basis of $\mathcal{L}^{m-1}(E_2 \times \cdots \times E_m)$; the result follows by induction.

**Corollary 2:** If $E_1, \ldots, E_m$ are reflexive Banach spaces with Schauder bases, then $\mathcal{L}^m(E_1 \times \cdots \times E_m)$ is reflexive if and only if it has monomial basis.

Proof: Holub [17] proved that if $E$ and $F$ are reflexive Banach spaces with bases, then $\mathcal{L}(E, F)$ is reflexive if and only if $\mathcal{L}(E, F) = \mathcal{K}(E, F)$. Holub’s result and Theorem 1 yield the proof.

**Remark:** Given $E_1, \ldots, E_m$ such that $\mathcal{L}^m(E_1 \times \cdots \times E_m)$ has a monomial basis, one may wonder if it has a non-monomial basis as well. Another result of Holub [17] is that if $X$ and $Y$ are reflexive Banach spaces with basis and their projective tensor product is not reflexive, then it has a non-tensor product basis. If any of the spaces $E_1, \ldots, E_m$ is not reflexive, neither is $\mathcal{L}^m(E_1 \times \cdots \times E_m)$ for this contains copies of $E_i'$'s. By a result of Zippin [25] not all bases of $\mathcal{L}^m(E_1 \times \cdots \times E_m)$ can be boundedly complete; thus there is a non-monomial basis (all monomial bases, being dual, are boundedly complete).

§2. $m$-linear forms over spaces with upper or lower $p$-bounds.

In this section we apply Theorem 1 to spaces with bounds for sequences.

We begin by recalling the necessary definitions of lower and upper indexes of a Banach space $E$, introduced by Gonzalo and Jaramillo in [14]. A sequence $\{x_i\}$ in $E$ is said to have an upper $p$-estimate ($1 \leq p \leq \infty$) if there is a positive constant $C$ such that for every $n$-tuple of scalars $a_1, \ldots, a_n$,

$$\|\sum_{i=1}^n a_i x_i\| \leq C \left( \sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}}$$

This is equivalent [15] to the sequence being weakly $q$-summable where $\frac{1}{p} + \frac{1}{q} = 1$. 


A Banach space is said to have property $S_p$ if every weakly null semi-normalized basic sequence in $E$ has a subsequence with an upper $p$-estimate. Clearly property $S_p$ implies property $S_r$ for all $r \leq p$. The lower index of $E$ is then defined as

$$l(E) = \sup \{ p \geq 1 : E \text{ has property } S_p \}$$

In an analogous manner, but using lower $q$-estimates one obtains the definition of property $T_q$ (with $T_q$ implying $T_r$ for all larger $r$), and

$$u(E) = \inf \{ q \geq 1 : E \text{ has property } T_q \}$$

We refer to [14] for more on the Gonzalo-Jaramillo indexes, but wish to mention that for $1 < p < \infty$, $l(\ell^p) = u(\ell^p) = p$. It is well-known (that monomials form a Schauder basis in the space of $k$-homogeneous polynomials over $\ell^p$ if $k < p$, and that when $k \geq p$, $P^k(\ell^p)$ contains a copy of $\ell^\infty$. Theorems 2 and 3 generalize these facts. First, we require a lemma. The proof is modeled on the proof of Theorem 2.4 of [14].

**Lemma:** Let $E_1, \ldots, E_N$, and $F$ be Banach spaces such that

$$\frac{1}{l(E_1)} + \cdots + \frac{1}{l(E_N)} < \frac{1}{u(F)}.$$

Then every $A \in \mathcal{L}^N(E_1 \times \cdots \times E_N, F)$ is weakly sequentially continuous.

**Proof:** Choose $p_1, \ldots, p_N, q$ so that $E_i$ has property $S_{p_i}$, $F$ has property $T_q$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}.$$

Take $\{x^n_i\} \subset E_i$ converging weakly but not in norm to 0 (if convergence is also strong for some $i$, there is nothing to prove). We may extract semi-normalized basic subsequences $\{x_{f_i(k)}^i\}$ with upper $p_i$-estimate. Thus for every $i = 1, \ldots, N$, the map $\ell^{p_i} \rightarrow E_i$ mapping $e_k$ to $x_{f_i(k)}^i$ is continuous. Given $A \in \mathcal{L}^N(E_1 \times \cdots \times E_N, F)$ define $\Phi \in \mathcal{L}^N(\ell^{p_1} \times \cdots \times \ell^{p_N}, F)$ by

$$\Phi(e_{i_1}, \ldots, e_{i_N}) = A(x_{f_1(i_1)}^1, \ldots, x_{f_N(i_N)}^N)$$

We shall see that $y_k = A(x_{f_1(k)}^1, \ldots, x_{f_N(k)}^N)$ is a sequence with an upper $(\frac{1}{p_1} + \cdots + \frac{1}{p_N})^{-1}$-estimate. For this, it will be sufficient [15] to verify that $(y_k)$ is weakly $s$-summable, where

$$s = \left[1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_N}\right)\right]^{-1}.$$

But given any $\gamma \in F^*$, $\gamma \circ \Phi \in \mathcal{L}^N(\ell^{p_1} \times \cdots \times \ell^{p_N})$, and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1 \leq 1.$$
so by [24], we obtain that $\gamma(y_k)$ is $s$-summable. Since $q < (\frac{1}{p_1} + \cdots + \frac{1}{p_N})^{-1}$, no subsequence of $(y_k)$ can have a lower $q$-estimate. But if some subsequence of $(y_k)$ were bounded from below in norm, since $F$ has property $T_q$ we could extract a subsequence with a lower $q$-estimate. Thus $y_k$ tend to 0 in norm. Since this can be done with any subsequence of $\{x_n^i\}$, we obtain that $A$ is weakly sequentially continuous at 0.

Suppose now that $\{x_n^i\}$ tend weakly to $x^i$. We will see by induction on $N$ that $A(x_n^1, \ldots, x_n^N)$ converges to $A(x^1, \ldots, x^N)$ in norm. For $N = 1$ this is a consequence of the weak sequential continuity at 0 and the linearity of $A$. Note that our hypothesis $(\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q})$ and the first part of the proof allow the weak sequential continuity of the $k$-linear functions ($k < N$) obtained by fixing some variables in $A$. Now $A(x_n^1 - x^1, \ldots, x_n^N - x^N)$ may be expanded to a sum of $2^{N-1}$ terms ($2^{N-1}$ added and $2^{N-1}$ subtracted). By adding and subtracting $A(x^1, \ldots, x^N)$ $2^{N-2}$ times we obtain

$$A(x_n^1 - x^1, \ldots, x_n^N - x^N) = A(x_n^1, \ldots, x_n^N) - A(x^1, \ldots, x^N)$$

$$+ \sum_{1} A(\ldots, x^1_i, \ldots, x_n^j, \ldots) - A(x^1, \ldots, x^N)$$

$$+ \sum_{2} A(x^1, \ldots, x^N) - A(\ldots, x^1_i, \ldots, x_n^j, \ldots)$$

where if $N$ is odd we have $2^{N-2}$ terms in $\sum_1$ and $2^{N-2}$ in $\sum_2$, whereas if $N$ is even we have $2^{N-2} - 1$ in $\sum_1$ and $2^{N-2} + 1$ in $\sum_2$. In any case,

$$\|A(x_n^1, \ldots, x_n^N) - A(x^1, \ldots, x^N)\|$$

$$\leq \|A(x_n^1 - x^1, \ldots, x_n^N - x^N)\| + \sum \|A(\ldots, x^1_i, \ldots, x_n^j, \ldots) - A(x^1, \ldots, x^N)\|$$

which tends to 0 by our inductive hypothesis and the weak sequential continuity of $A$ at 0.

**Theorem 2:** Let $E_1, \ldots, E_m$ be Banach spaces with shrinking bases $\{e_n^i\}$ and such that for some $j$

$$\frac{1}{l(E_1)} + \cdots + \frac{1}{l(E_j)} + \cdots + \frac{1}{l(E_m)} < \frac{1}{u(E_j^*)}$$

Then the monomials $B_{i_1, \ldots, i_m} = e_{i_1}^{1*} \cdots e_{i_m}^{m*}$ form a Schauder basis of $\mathcal{L}^m(E_1 \times \cdots \times E_m)$.

Proof: Just apply the Lemma and equivalences (iv) and (viii) of Theorem 1.

**Corollary:** Let $E_1, \ldots, E_m$ be reflexive Banach spaces with Schauder bases and such that for some $j$

$$\frac{1}{l(E_1)} + \cdots + \frac{1}{l(E_j)} + \cdots + \frac{1}{l(E_m)} < \frac{1}{u(E_j^*)}$$
Then \( \mathcal{L}^m(E_1 \times \cdots \times E_m) \) is reflexive.

Proof: The proof is as that of Corollary 1 of \( \S 1 \).

**Theorem 3:** Let \( E_1, \ldots, E_m \) be Banach spaces with unconditional shrinking bases, and such that for all \( j \)

\[
\frac{1}{u(E_1)} + \cdots + \frac{1}{u(E_j)} + \cdots + \frac{1}{u(E_m)} > \frac{1}{l(E_j^*)}
\]

Then \( \mathcal{L}^m(E_1 \times \cdots \times E_m) \) contains a copy of \( \ell^\infty \).

Proof: None of the \( E_i \)'s contain a copy of \( \ell^1 \), for all have shrinking bases. Thus they either contain copies of \( \ell_0 \) or are reflexive. Those which contain copies of \( \ell_0 \) have infinite upper Gonzalo-Jaramillo index. Therefore the inequality in our hypothesis forces at least one the \( E_i \)'s to be reflexive. Say \( E_m \) is reflexive. We have

\[
\frac{1}{u(E_1)} + \cdots + \frac{1}{u(E_{m-1})} > \frac{1}{l(E_m^*)}.
\]

Choose \( p, q_1, \ldots, q_{m-1} \) such that \( E_i \) has property \( T_{q_i} \) for \( i \leq m-1 \), \( E_m^* \) has property \( S_p \), and \( \frac{1}{q_1} + \cdots + \frac{1}{q_{m-1}} = \frac{1}{p} \). Given the unconditional shrinking bases \( (e_i^j)_i \) of \( E_j \), by normalizing when necessary, we obtain weakly null seminormalized basic sequences from which we extract subsequences (with the same subindexes), such that \( (e_i^j)_i \) has a lower \( q_j \)-estimate. Since \( E_m \) is reflexive, \( (e_i^{m*})_i \) is shrinking; thus in the same manner as above we extract a subsequence \( (e_i^{m*})_i \) with an upper \( p \)-estimate.

For each \( i \in \mathbb{N} \) define \( T_i \in \mathcal{L}^m(E_1 \times \cdots \times E_m) \) by setting

\[
T_i(x^1, \ldots, x^m) = x_{i_1}^1 \cdots x_{i_n}^m.
\]

In other words, \( T_i = e_{i_1}^{1*} \cdots e_{i_n}^{m*} \). Given scalars \( a_1, \ldots, a_n \),

\[
\left\| \sum_{i=1}^{n} a_i T_i \right\| = \sup_{\|x^j\| \leq 1} \left| \sum_{i=1}^{n} a_i x_{i_{n_i}}^1 \cdots x_{i_{n_i}}^m \right|
\]

\[
= \sup_{\|x^j\| \leq 1} \left\| \sum_{i=1}^{n} a_i x_{i_{n_i}}^1 \cdots x_{i_{n_i}}^{m-1} e_{n_i}^{m*} \right\|
\]

\[
\leq D \sup_{\|x^j\| \leq 1} \left( \sum_{i=1}^{n} |a_i x_{i_{n_i}}^1 \cdots x_{i_{n_i}}^{m-1}|^p \right)^{\frac{1}{p}}
\]

\[
\leq \sup_i |a_i| D \sup_{\|x^j\| \leq 1} \left( \sum_{i=1}^{n} |x_{i_{n_i}}^1 \cdots x_{i_{n_i}}^{m-1}|^p \right)^{\frac{1}{p}}
\]

\[
\leq \|a\|_\infty D \sup_{\|x^j\| \leq 1} \left( \sum_{i=1}^{n} |x_{i_{n_i}}^{1}|^{q_1} \cdots \sum_{i=1}^{n} |x_{i_{n_i}}^{m-1}|^{q_{m-1}} \right)^{\frac{1}{q_{m-1}}}
\]

\[
\leq C_1 \cdots C_{m-1} D \|a\|_\infty \sup_{\|x^j\| \leq 1} \left\| \sum_{i=1}^{n} x_{i_{n_i}}^{1} e_{n_i}^{1} \right\| \cdots \left\| \sum_{i=1}^{n} x_{i_{n_i}}^{m-1} e_{n_i}^{m-1} \right\|
\]

\[
\leq K C_1 \cdots C_{m-1} D \|a\|_\infty
\]
where $K$ is the maximum unconditional basis constant of the bases of the $E_i$'s. Thus
\[
\left\| \sum_{i=1}^{n} a_i T_i \right\| \leq C \|a\|_{\infty}.
\]
On the other hand, for each $k \in \mathbb{N}$,
\[
|a_k| = \left| \left( \sum_{i=1}^{n} a_i T_i \right)(e_{n_k}^1, \ldots, e_{n_k}^m) \right| \\
\leq \left\| \sum_{i=1}^{n} a_i T_i \right\| \|e_{n_k}^1\| \cdots \|e_{n_k}^m\|
\]
so
\[
\|a\|_{\infty} = \sup_{k} |a_k| \leq \sup_{k} \left\| e_{n_k}^1 \right\| \cdots \left\| e_{n_k}^m \right\| \left\| \sum_{i=1}^{n} a_i T_i \right\|.
\]
Therefore there are constants $C'$ such that
\[
C' \|a\|_{\infty} \leq \left\| \sum_{i=1}^{n} a_i T_i \right\| \leq C \|a\|_{\infty},
\]
and the closed subspace of $L^m(E_1 \times \cdots \times E_m)$ spanned by the $T_i$'s is isomorphic to $c_0$. Being a dual space, $L^m(E_1 \times \cdots \times E_m)$ must also contain a copy of $\ell^\infty$.

Note that the proof of the theorem remains valid as long as one has the inequality for a given $j$, with a reflexive $E_j$. Also, it is not necessary for the basis of this $E_j$ to be unconditional. An example in which the inequality of Theorem 3 is verified by some but not all $j$'s is provided by $L^2(c_0 \times \ell^2)$. This space has a basis by Theorem 2.

§3. Polynomials over spaces with upper or lower $p$-bounds.

Here we apply the results of the previous sections to spaces of polynomials over a Banach space $E$. The set $D = \{(i_1, \ldots, i_m) \in \mathbb{N}^m : i_1 \geq i_2 \geq \ldots \geq i_m\}$ is given the ordering induced by the square ordering of $\mathbb{N}^m$. For each $(i_1, \ldots, i_m) \in D$ we will note by $P_{i_1, \ldots, i_m}$ the $m$-homogeneous polynomial
\[
P_{i_1, \ldots, i_m}(x) = B_{i_1, \ldots, i_m}(x, \ldots, x)
\]
and by the notation $\{P_{i_1, \ldots, i_m}\}$ we mean the sequence $\{P_{i_1, \ldots, i_m}\}_{(i_1, \ldots, i_m) \in D}$ ordered as above. We then have the following proposition.

**Proposition:** Let $E$ be a Banach space with Schauder basis such that $\{B_{i_1, \ldots, i_m}\}$ is a basis of $L^m(E)$. Then $\{P_{i_1, \ldots, i_m}\}$ is a basis of $P^m(E)$.

**Proof:** Clearly, if $\{B_{i_1, \ldots, i_m}\}$ is a basis of $L^m(E)$ then $\{P_{i_1, \ldots, i_m}\}$ spans a dense subspace in $P^m(E)$. Since $B_{i_1, \ldots, i_m}$ are the coordinate functionals associated with
the basis of \( \hat{\otimes}_{m,s}E \) (the \( m \)-fold symmetric projective tensor product), they form a basic sequence. Consequently, \( \{P_{i_1,\ldots,i_m}\} \) is a basis of \( \mathcal{P}^m(E) \).

As simple consequences of this Proposition, we have the following analogues of Theorem 2 and its Corollary:

**Theorem 4:** Let \( E \) be a Banach space with shrinking basis such that

\[
(m - 1)u(E^*) < l(E)
\]

Then \( \{P_{i_1,\ldots,i_m}\} \) is a basis of \( \mathcal{P}^m(E) \).

**Corollary:** Let \( E \) be a reflexive Banach space with a basis such that

\[
(m - 1)u(E^*) < l(E)
\]

Then \( \mathcal{P}^m(E) \) is reflexive.

Theorem 3 can also be proved for spaces of homogeneous polynomials. As we’ve mentioned above, when \( m > p \) the space of \( m \)-homogeneous polynomials \( \mathcal{P}^m(\ell^p) \) contains a copy of \( \ell^\infty \) (see for example [1], [7], [13]). The following theorem generalizes that result.

**Theorem 5:** Let \( E \) be a Banach space with unconditional shrinking basis such that

\[
(m - 1)l(E^*) > u(E)
\]

Then \( \mathcal{P}^m(E) \) contains a copy of \( \ell^\infty \).

**Proof:** If we put \( E_1 = E_2 = \ldots = E_m = E \) and \( q_1 = \ldots = q_{m-1} = q \) in the proof of Theorem 3 we obtain \( T_i = e_{n_i}^* \cdots e_{n_i}^* \in \mathcal{L}^m(E) \), for the \( T_i \)'s are symmetric. Thus \( c_0 \subset [T_i] \subset \mathcal{L}^m_l(E) \cong \mathcal{P}^m(E) \). Being a dual, \( \mathcal{P}^m(E) \) contains \( \ell^\infty \).

§4. Examples and applications.

In this section we apply our results to several concrete Banach space examples. In several cases, and particularly for the polynomial examples, these results are known, with different proofs. One should consult, for example [5], [2], [3], [7], [12], [13]. The Gonzalo-Jaramillo indexes used can be found in [14].

Example 1: \( \ell^p \) spaces (\( 1 < p < \infty \)).

In this case one has \( l(\ell^p) = u(\ell^p) = p \). Thus the following conditions are all equivalent:

a) \( \frac{1}{p_1} + \ldots + \frac{1}{p_m} < 1 \).

b) \( \mathcal{L}^m(\ell^{p_1} \times \ldots \times \ell^{p_m}) \) is reflexive.

c) \( \mathcal{L}^m(\ell^{p_1} \times \ldots \times \ell^{p_m}) \) has a monomial basis.
d) Every operator from $\ell^{p_1}$ to $\mathcal{L}^{m-1}(\ell^{p_2} \times \cdots \times \ell^{p_m})$ is compact.

To see d) $\Rightarrow$ a), note that if 

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} \geq 1$$

the operator $T$ defined by

$$T(x^1)(x^2, \ldots, x^m) = \sum_{n=1}^{\infty} x^1_n \cdots x^m_n$$

is not compact. As a consequence of the Corollary 1 of §1, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} \geq 1$ if and only if $\mathcal{L}^m(\ell^{p_1} \times \cdots \times \ell^{p_m})$ contains a copy of $\ell^\infty$. In particular, if $p > m$, $\mathcal{P}^m(\ell^p)$ has monomial basis and is reflexive, while if $p \leq m$, $\mathcal{P}^m(\ell^p)$ contains $\ell^\infty$.

Example 2: reflexive Orlicz spaces.

If $\ell^M$ is a reflexive Orlicz space, it has an unconditional shrinking basis [19] and $l(\ell^M) = \alpha_M$, $u(\ell^M) = \beta_M$, the upper and lower Boyd indexes. Thus we have

i) If $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_m}} < 1$, then $\mathcal{L}^m(\ell^{M_1} \times \cdots \times \ell^{M_m})$ has monomial basis and is reflexive.

ii) If $\frac{1}{\beta_{M_1}} + \cdots + \frac{1}{\beta_{M_m}} > 1$, then $\mathcal{L}^m(\ell^{M_1} \times \cdots \times \ell^{M_m})$ contains $\ell^\infty$.

For polynomials; if $\alpha_M > m$, $\mathcal{P}^m(\ell^M)$ has monomial basis and is reflexive, while if $\beta_M < m$, $\mathcal{P}^m(\ell^M)$ contains $\ell^\infty$.

Example 3: $c_0$ and Tsirelson space.

In these cases $l(c_0) = u(c_0) = \infty$, $l(\ell^1) = \infty$, $u(\ell^1) = 1$; and $l(T) = u(T) = 1$, $l(T^*) = u(T^*) = \infty$ (we denote the original Tsirelson space by $T$). Since $c_0$, $T$, and $T^*$ have shrinking bases, and $T$ is reflexive (and its basis is unconditional), we have

i) $\mathcal{L}^m(c_0 \times \cdots \times c_0)$ has monomial basis for all $m$.

ii) $\mathcal{L}^m(T^* \times \cdots \times T^*)$ has monomial basis and is reflexive for all $m$.

iii) $\mathcal{L}^m(T \times \cdots \times T)$ contains a copy of $\ell^\infty$ for all $m \geq 2$.

and analogous results hold for the corresponding spaces of homogeneous polynomials.

Example 4: $L^p[0,1]$, for $1 < p < \infty$.

Here $l(L^p[0,1]) = \min\{2,p\}$, and $u(L^p[0,1]) = \max\{2,p\}$. For $m = 2$, $\mathcal{L}^m(L^p[0,1] \times \cdots \times L^p[0,1])$ never verifies the inequalities of theorems 2 and 3. If $m > 2$, the inequality of theorem 3 may or may not be verified. In any case, it is easily seen...
that \( L^m(L^{p_1}[0,1] \times \cdots \times L^{p_m}[0,1]) \) always contains a copy of \( \ell^\infty \), for \( \ell^2 \) is isomorphic to a complemented subspace of \( L^p[0,1] \). In the same manner, the space of homogeneous polynomials \( \mathcal{P}^m(L^p[0,1]) \) \((m > 1)\) contains a copy of \( \ell^\infty \).

An application to tensor products:

If \( E_1, \ldots, E_m \) are reflexive Banach spaces with a basis, we have already mentioned (Corollary 2 of §1) as a consequence of a result of Holub [17], that \( L^m(E_1 \times \cdots \times E_m) \) has a basis if and only if it is reflexive. In that case we know from Theorem 1 that every linear operator from \( E_1 \) to \( L^{m-1}(E_2 \times \cdots \times E_m) \) is compact. By transposing we then obtain that every linear operator from \( E_2 \hat{\otimes} \cdots \hat{\otimes} E_m \) to \( E_1^* \) is compact. If we relate this to the Gonzalo-Jaramillo indexes we have the following result which generalizes Pitt’s theorem [21] and a result of Pelczynski [20].

**Proposition:** Let \( E_1, \ldots, E_m \) be reflexive Banach spaces with bases and such that for some \( j \)

\[
\frac{1}{l(E_1)} + \cdots + \frac{1}{l(E_j)} + \cdots + \frac{1}{l(E_m)} < \frac{1}{u(E_j^*)}
\]

Then every linear operator from \( E_1 \hat{\otimes} \cdots \hat{\otimes} E_m \) to \( E_j^* \) is compact (here the tensor products exclude \( E_j \)).

In the case of \( \ell^p \) spaces the converse is also true. Thus we have the following result which was communicated to us by T. W. Gamelin [12]. We have since learned that it has also been proved by R. Alencar and K. Floret [3].

**Theorem (Alencar-Floret/Gamelin):** Every \( T \in \mathcal{L}(\ell^{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell^{p_m}, \ell^r) \) is compact if and only if \( \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{r} \).

We end with some problems which we believe are still open.

1. Is it possible for \( L^m(E_1 \times \cdots \times E_m) \) to have neither monomial basis nor a subspace isomorphic to \( \ell^\infty \)? \((E_1, \ldots, E_m \) with shrinking but non-unconditional bases).

2. We have observed that if \( \{B_{i_1, \ldots, i_m}\} \) is a Schauder basis of \( L^m(E) \), then \( \{P_{i_1, \ldots, i_m}\} \) is a Schauder basis of \( \mathcal{P}^m(E) \). Is the reciprocal true?

3. In the construction of monomial bases the “square ordering” seems all-important. Can a monomial basis be unconditional?

4. For \( m > 1 \) and \( E \) infinite dimensional, can \( \mathcal{P}^m(E) \) and \( \mathcal{P}^m(E^*) \) both be reflexive? Consideration of the Gonzalo-Jaramillo indexes seems to suggest a negative answer, but perhaps some stronger hypothesis is required.
References


